[09-11-14-T11]

Essential Facts for trigonometry

The number of facts used in basic trigonometry may seem daunting. This need not be the case. Most are immediate consequences of a few definitions and are obvious when one considers the unit circle. The rest follow, almost immediately, once an addition theorem is proved; we have chosen [12] for this purpose. If one understands the proofs of the identities, then there is really very little that one must commit to memory. But, the understanding must be so good, that one can produce any of these identities very quickly. Such understanding is best acquired by drawing a unit circle, stating the definitions and the addition theorem [12], then actually deriving, several times, the sequence of identities below.

The unit circle is a circle center $O(0, 0)$, radius 1. A point $P(u, v)$ on the unit circle has coordinates (u, v).

In radian measure, an arc of length x on the unit circle subtends a central angle size x. Thus, we may speak of angle x or arc length x.

Every real number x can be represented by an arc of length x on the unit circle starting at the point $(1, 0)$ with the counter-clockwise direction having the positive sense.

Definition

[3] $\sin x = v$ $\qquad \csc x = \frac{1}{\sin x}$

$\cos x = u$ $\qquad \sec x = \frac{1}{\cos x}$

$\tan x = \frac{v}{u} = \frac{\sin x}{\cos x}$ $\qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

If one keeps in mind these definitions, a number of obvious but important facts may be obtained by considering the characteristics of the unit circle.

Consequences of these definitions

Since u and v cannot be less than -1 nor greater than 1,

[4]	$-1 \leq \sin x \leq 1$	$-1 \leq \cos x \leq 1$

Applying the theorem of Pythagoras to u, v and OP,

[5]	$\sin^2 x + \cos^2 x = 1$

Divide [5] by $\cos^2 x$,

[6]	$\tan^2 x + 1 = \sec^2 x$

Divide [5] by $\sin^2 x$,

[7]	$1 + \cot^2 x = \csc^2 x$

By the periodicity of the circle,

[8]	$\sin(x + 2 n \pi) = \sin x$;	$\cos(x + 2 n \pi) = \cos x$;	$\tan(x + n \pi) = \tan x$

Keeping in mind the definitions of [3], by symmetry,

[9]	$\sin(-x) = -\sin x$;	$\cos(-x) = \cos(x)$;	$\tan(-x) = -\tan x$

[10]	$\sin(\frac{\pi}{2} - x) = \cos x$	$\sin(\frac{\pi}{2} + x) = \cos x$
	$\cos(\frac{\pi}{2} - x) = \sin x$	$\cos(\frac{\pi}{2} + x) = -\sin x$
	$\tan(\frac{\pi}{2} - x) = \cot x$	$\tan(\frac{\pi}{2} + x) = -\cot x$

[11]	$\sin(\pi - x) = \sin x$	$\sin(\pi + x) = -\sin x$
	$\cos(\pi - x) = -\cos x$	$\cos(\pi + x) = -\cos x$
	$\tan(\pi - x) = -\tan x$	$\tan(\pi + x) = \tan x$

Consequences of addition theorem

Identity [12] is an addition theorem. Its proof is at the end of this section.

[12] $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Substitution of $-y$ for y in [12],

[13] $\cos(x - y) = \cos x \cos y + \sin x \sin y$

Write $\sin(x + y)$ as $\cos(\frac{\pi}{2} - (x + y)) \Longleftrightarrow \cos((\frac{\pi}{2} - x) - y)$ and use [13],

[14] $\sin(x + y) = \sin x \cos y + \cos x \sin y$

Substitution of $-y$ for y in [14],

[15] $\sin(x - y) = \sin x \cos y - \cos x \sin y$

Divide [12] by [14],

[16] $\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

Divide [13] by [15],

[17]	$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \, \tan y}$

■ **Double angle identities**

Substitution of x for y in [12],

[18]	$\cos 2x = \cos^2 x - \sin^2 x$

Substitution of x for y in [14],

[19]	$\sin 2x = 2 \sin x \; \cos x$

Use $\sin^2 x + \cos^2 = 1$ with [18],

[20]	$\cos 2x = 2 \cos^2 x - 1$

Use $\sin^2 x + \cos^2 = 1$ with [18],

[21]	$\cos 2x = 1 - 2 \sin^2 x$

Substitution of x for y in [16],

[22]	$\tan 2x = \frac{2 \tan x}{1-\tan^2 x}$

■ Half angle identities

Solve [20] for $\cos^2 x$,

$$[23] \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Solve [21] for $\sin^2 x$,

$$[24] \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Divide [24] by [23]

$$[25] \quad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Add [12] and [13],

■ Product identities

$$[26] \quad \cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

Add [14] and [15],

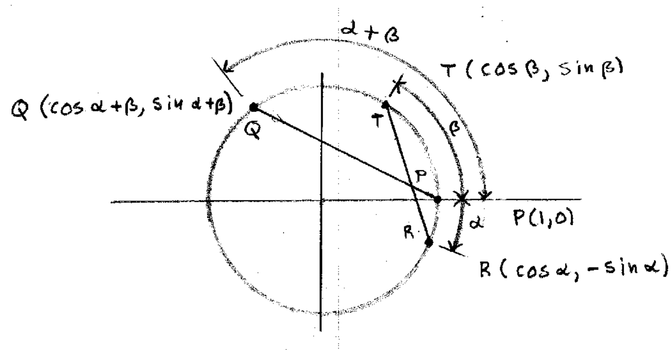
$$[27] \quad \sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

Subtract [12] from [13]

$$[28] \quad \sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

■ **The proofs of [12], [14], [16], and [26] are not as obvious as those of the other identities.**

To prove [12]



Assume as in the figure above that the sum of an arc of length α and an arc of length β is an arc of length $\alpha + \beta$.

Since $\overset{\text{arc}}{\text{PQ}} = \overset{\text{arc}}{\text{RT}}$, the lengths chords PQ and RT are equal.

PQ^2
$= [\cos(\alpha+\beta) - 1]^2 + [\sin(\alpha+\beta) - 0]^2$
$= \cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta)$
$= 2 - 2\cos(\alpha+\beta)$

RT^2
$= [\cos\beta - \cos\alpha]^2 + [\sin\beta + \sin\alpha]^2$
$= \cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta + \sin^2\alpha$
$= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$

Since PQ = RT

PQ = RT
$\Longleftrightarrow 2 - 2\cos(\alpha+\beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$
$\Longleftrightarrow -2\cos(\alpha+\beta) = -2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$
$\Longleftrightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Therefore,

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

□

To get [14]

$$\sin(x+y)=\cos(\tfrac{\pi}{2}-(x+y))=\cos((\tfrac{\pi}{2}-x)-y)=\cos(\tfrac{\pi}{2}-x)\cos y+\sin(\tfrac{\pi}{2}-x)\ \sin y$$
$$=\sin x\ \cos y+\cos x\ \sin y$$

To get [16]

$$\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin\ x\ \cos y+\cos\ x\ \sin y}{\cos\ x\ \cos y-\sin\ x\ \sin y}=\frac{\sin\ x\ \cos y+\cos\ x\ \sin y}{\cos\ x\ \cos y-\sin\ x\ \sin y}\cdot\frac{\frac{1}{\cos x\ \cos y}}{\frac{1}{\cos x\ \cos y}}$$
$$=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

To get [26]

$$\cos(x+y)+\cos(x-y)=(\cos\ x\ \cos y-\sin\ x\ \sin y)+(\cos x\cos y+\sin x\sin y)$$
$$\Longleftrightarrow\cos(x+y)+\cos(x-y)=2\cos\ x\ \cos y$$
$$\Longleftrightarrow\cos\ x\ \cos y=\tfrac{1}{2}\,(\cos(x+y)+\cos(x-y))$$